Problem Solving 9

Lecture 20 May 30 , 2021

The following questions are the 6 questions of some semi-advanced Math Olympiad competition.

The exam was held in two consecutive days, with 3 questions each day for 4.5 hours (so 1.5 hour per question).

So you may put yourself in such an exam condition and think about these problems as if you are taking the exam.

- Day 1, Q1. Suppose p is a prime number and n is a natural number, so that $1 + np$ is a perfect square. Prove that we can write $n + 1$ as a sum of p perfect squares.

- Day 1, Q2. Suppose all the angles of the triangle ABC are acute. We build 3 triangles A’BC, AB’C, and ABC’ on the edges of ABC so that A’,B’, and C’ are outside ABC and

- $\angle B'AC = \angle C'BA = \angle A'BC = 30$
- $\angle B'CA = \angle C'AB = \angle A'CB = 60$

If M is the middle point of BC, show that B’M is perpendicular to A’C’.

- Day 1, Q3. Find all natural numbers N so that we can put N equal squares on the plane in a way that

- all the edges are parallel to x-axis or y-axis
- The resulting configuration is symmetric with respect to at least 3 lines (i.e. symmetric under reflection)

- Day 2, Q4. Find all the polynomials $P(x)$ with real coefficients that satisfy the equation

$$P(2P(x)) = 2P(P(x)) + 2P(x)^2$$

- Day 2, Q5. In the triangle ABC, with |AB|>|AC|, the angle bisectors of B and C intersect the edges AC and AB at P and Q, respectively. Let M denote the intersection points of the BP and CQ. If MP=MQ, what is the angle A?

- Day 2, Q6. Let $a_1, a_2, a_3, \ldots$ be a sequence of non-negative integers such that only finitely many of them are nonzero. In each turn, if possible, we can perform one of the following changes

1) If a_n and a_{n+1} are nonzero for some $n \geq 1$, we replace a_n and a_{n+1} with $a_n - 1$ and $a_{n+1} - 1$, and we replace a_{n+2} with $a_{n+2} + 1$

2) If $a_n > 1$ for some $n \geq 3$, we replace a_n with $a_n - 2$, a_{n+1} with $a_{n+1} + 1$, and a_{n-2} with $a_{n-2} + 1$.

- Prove that starting from any initial sequence, after finitely many steps, we reach a situation that none of the actions above is possible.

- Suppose that the starting sequence satisfies $a_1 = a_2 = a_3 = \cdots = a_n = 1$ for some $n \geq 1$ and the rest of them are zero. Prove that a_m will remain zero for all $m > n + 1$.